

On invariants of second-order ordinary differential equations $y'' = f(x, y, y')$ via point transformations

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Abstract

Bagderina [1] solved the equivalence problem for a family of scalar second-order ordinary differential equations (ODEs), with cubic nonlinearity in the first-order derivative, via point transformations. However, the question is open for the general class $y'' = f(x, y, y')$ which is not cubic in the first-order derivative. We utilize Lie's infinitesimal method to study the differential invariants of this general class under an arbitrary point equivalence transformations. All fifth order differential invariants and the invariant differentiation operators are determined. As an application, invariant description of all the canonical forms in the complex plane for second-order ODEs $y'' = f(x, y, y')$ where both of the two Tressé relative invariants are non-zero is provided.

Keywords: Lie's infinitesimal method, differential invariants, second order ODEs, equivalence problem, point transformations, canonical forms, Lie symmetries.

1 Introduction

Lie's group classification of ODEs shows that the second-order equations can possess one, two, three or eight infinitesimal symmetries. According to Lie's classification [2] in the complex domain, any second order ODE

$$y'' = f(x, y, y'), \quad (1.1)$$

is obtained by a change of variables from one of eight canonical forms. The equations with eight symmetries and only these equations can be linearized by a change of variables. The initial seminal studies of scalar second-order ODEs which are linearizable by means of point transformations are due to Lie [2] and Tressé [3]. They showed that the latter equations are at most cubic in the first derivative and gave a convenient invariant description of all linearizable equations. Mahomed and Leach [4] proved that Lie linearization conditions are equivalent to the vanishing of the Tressé relative invariants (1.2) as stated in the next theorem

Theorem 1.1. [5] *The equation $y'' = f(x, y, y')$ is equivalent to the normal form $y'' = 0$ with eight symmetries under **point transformations** if and only if the Tressé relative invariants*

$$\begin{aligned} I_1 &= f_{y',y',y',y'} \\ I_2 &= \dot{D}_x^2 f_{y',y'} - 4\dot{D}_x f_{y,y'} - 3f_y f_{y',y'} + 6f_{y,y} + f_{y'} \left(4f_{y,y'} - \dot{D}_x f_{y',y'} \right) \end{aligned} \quad (1.2)$$

both vanish identically. Where $\dot{D}_x = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + f \frac{\partial}{\partial y'}$.

Regarding the equivalence of the second-order differential equations to the normal form $y'' = 0$ via contact transformations, it is well known that all second-order differential equations have $y'' = 0$ as the sole equivalence class.

The linearization problem is a particular case of the equivalence problem. For the general theory of the equivalence problem including algorithms of construction of differential

invariants, the interested reader is referred to [6, 9]. Ibragimov [10, 12] developed a simple method for constructing invariants of families of linear and nonlinear differential equations admitting infinite equivalence transformation groups. Lie's infinitesimal method was applied to solve the equivalence problem for several linear and nonlinear differential equations [13, 14, 15, 16, 17, 18, 19, 20, 21]. Cartan's equivalence method [6, 22] is another systematic approach to solve the equivalence problem for differential equations.

By using Lie's infinitesimal method, Bagderina [1] solved the equivalence problem of second-order ODEs which are at most cubic in the first-order derivative ($I_1 = 0$)

$$y'' = a(x, y)y'^3 + b(x, y)y'^2 + c(x, y)y' + d(x, y) \quad (1.3)$$

with respect to the group of point equivalence transformations

$$\bar{x} = \phi(x, y), \bar{y} = \psi(x, y). \quad (1.4)$$

As an extension, in this paper, we use Lie's infinitesimal method to study the differential invariants of the second-order ODEs (1.1) which are not cubic in the first-order derivative ($I_1 \neq 0$) with respect to the group of point equivalence transformations. The motivation of this study is finding invariant description of the canonical forms for second-order ODEs in the complex plane [6] which are not cubic in the first-order derivative.

Invariant description of the canonical forms for second-order ODEs in the complex plane with three infinitesimal symmetries was given in [8, 15] where they presented the candidates for all four types and then they studied these candidates. In this paper, invariant description of all the canonical forms in the complex plane for second-order ODEs $y'' = f(x, y, y')$ where both of the two Tressé relative invariants (1.2) are non-zero is provided.

The structure of the paper is as follows. In the next section, we give a short description of Lie's infinitesimal method to find the differential invariants and invariant differentiation

operators of the class of ODEs (1.1) with respect to the group of point equivalence transformations (1.4). In Section 3, using the methods described in Section 2, the infinitesimal point equivalence transformations are recovered. In Section 4, we find the fifth-order differential invariants and invariant differentiation operators of the class of ODEs (1.1), which are not cubic in the first-order derivative, under point equivalence transformations. In Section 5, invariant description of all the canonical forms in the complex plane [6] for second-order ODEs $y'' = f(x, y, y')$ where both of the two Tressé relative invariants (1.2) are non-zero is provided. Finally, the conclusion is presented.

Throughout this paper, we use the notation $A = [a_1, a_2, \dots, a_n]$ to express any differential operator $A = \sum_{j=1}^n a_j \frac{\partial}{\partial b_j}$. Also, we denote y' by p .

2 Lie's infinitesimal method

In this section, we briefly describe the Lie method used to derive differential invariants using point equivalence transformations.

Consider the k th-order system of PDEs of n independent variables $x = (x^1, x^2, \dots, x^n)$ and m dependent variables $u = (u^1, u^2, \dots, u^m)$

$$E_\alpha(x, u, \dots, u_{(k)}) = 0, \quad \alpha = 1, \dots, m, \quad (2.5)$$

where $u_{(1)}, u_{(2)}, \dots, u_{(k)}$ denote the collections of all first, second, ..., k th-order partial derivatives, i.e., $u_i^\alpha = D_i(u^\alpha)$, $u_{ij}^\alpha = D_j D_i(u^\alpha), \dots$, respectively, in which the total differentiation operator with respect to x^i is

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \dots, \quad i = 1, \dots, n, \quad (2.6)$$

with the summation convention adopted for repeated indices.

Definition 2.1. *The Lie-Bäcklund operator is*

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} \quad \xi^i, \eta^\alpha \in A, \quad (2.7)$$

where A is the space of *differential functions*.

The operator (2.7) is an abbreviated form of the infinite formal sum

$$\begin{aligned} X^{(s)} &= \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} \zeta_{i_1 i_2 \dots i_s}^\alpha \frac{\partial}{\partial u_{i_1 i_2 \dots i_s}^\alpha}, \\ &= \xi^i D_i + W^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} D_{i_1} \dots D_{i_s} (W^\alpha) \frac{\partial}{\partial u_{i_1 i_2 \dots i_s}^\alpha}, \end{aligned} \quad (2.8)$$

where the additional coefficients are determined uniquely by the prolongation formulae

$$\begin{aligned} \zeta_i^\alpha &= D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j) = D_i(W^\alpha) + \xi^j u_{ij}^\alpha, \\ \zeta_{i_1 \dots i_s}^\alpha &= D_{i_s}(\zeta_{i_1 \dots i_{s-1}}^\alpha) - u_{j i_1 \dots i_{s-1}}^\alpha D_{i_s}(\xi^j) = D_{i_1} \dots D_{i_s}(W^\alpha) + \xi^j u_{j i_1 \dots i_s}^\alpha, \quad s > 1, \end{aligned} \quad (2.9)$$

in which W^α is the *Lie characteristic function*

$$W^\alpha = \eta^\alpha - \xi^j u_j^\alpha. \quad (2.10)$$

Definition 2.2. *The point equivalence transformation of a class of PDEs (2.5) is an invertible transformation of the independent and dependent variables of the form*

$$\bar{x} = \phi(x, u), \bar{u} = \psi(x, u), \quad (2.11)$$

that maps every equation of the class into an equation of the same family, viz.

$$E_\alpha(\bar{x}, \bar{u}, \dots, \bar{u}_{(k)}) = 0, \quad \alpha = 1, \dots, m. \quad (2.12)$$

In order to describe Lie's infinitesimal method for deriving differential invariants using point equivalence transformations, we use as example the class of equations (1.1). It is well-known that the point equivalence transformation

$$\bar{x} = \phi(x, y), \bar{y} = \psi(x, y), \quad (2.13)$$

maps (1.1) into the same family, viz.

$$\bar{y}'' = \bar{f}(\bar{x}, \bar{y}, \bar{y}'), \quad (2.14)$$

for arbitrary functions $\phi(x, y)$ and $\psi(x, y)$, where \bar{f} , in general, can be different from the original function f . The set of all equivalence transformations forms a group denoted by \mathcal{E} .

The standard procedure for Lie's infinitesimal invariance criterion [9] is implemented in the next section to recover the continuous group of point equivalence transformations (2.13) for the class of second-order ODEs (1.1) with the corresponding infinitesimal point equivalence transformation operator

$$Y = \xi(x, y)D_x + W\partial_y + D_x(W)\partial_p + \mu(x, y, p, f)\partial_f, \quad (2.15)$$

where $\xi(x, y)$ and $\eta(x, y)$ are arbitrary functions obtained from

$$\bar{x} = x + \epsilon \xi(x, y) + O(\epsilon^2) = \phi(x, y), \quad (2.16)$$

$$\bar{y} = y + \epsilon \eta(x, y) + O(\epsilon^2) = \psi(x, y), \quad (2.17)$$

and

$$\mu = \dot{D}_x^2(W) + \xi(x, y)\dot{D}_x f, \quad (2.18)$$

with $W = \eta - \xi p$ and $\dot{D}_x = \frac{\partial}{\partial x} + p\frac{\partial}{\partial y} + f\frac{\partial}{\partial p}$.

Definition 2.3. *An invariant of a class of second-order ODEs (1.1) is a function of the form*

$$J = J(x, y, p, f), \quad (2.19)$$

which is invariant under the equivalence transformation (2.13).

Definition 2.4. *A differential invariant of order s of a class of second-order ODEs (1.1) is a function of the form*

$$J = J(x, y, p, f, f_{(1)}, f_{(2)}, \dots, f_{(s)}), \quad (2.20)$$

which is invariant under the equivalence transformation (2.13) where $f_{(1)}, f_{(2)}, \dots, f_{(s)}$ denote the collections of all first, second, ..., sth-order partial derivatives.

Definition 2.5. *An invariant system of order s of a class of second-order ODEs (1.1) is the system of the form $E_\alpha(x, y, p, f, f_{(1)}, f_{(2)}, \dots, f_{(s)}) = 0$, $\alpha = 1, \dots, m$ which satisfies the condition*

$$Y^{(s)} E_\alpha(x, y, p, f, f_{(1)}, f_{(2)}, \dots, f_{(s)}) = 0 \pmod{E_\alpha = 0, \alpha = 1, \dots, m}, \alpha = 1, \dots, m. \quad (2.21)$$

An invariant system with $\alpha = 1$ is called an invariant equation.

Now, according to the theory of invariants of infinite transformation groups [9], the invariant criterion

$$YJ(x, y, p, f) = 0, \quad (2.22)$$

should be split by means of the functions $\xi(x, y)$ and $\eta(x, y)$ and their derivatives. This gives rise to a homogeneous linear system of PDEs whose solution gives the required invariants.

It should be noted that since the generator Y contains arbitrary functions $\xi(x, y)$ and $\eta(x, y)$, the corresponding identity (2.22) leads to m linear PDEs for J , where m is the number of the arbitrary functions and their derivatives that appear in Y . We point out that these m PDEs are not necessarily linearly independent.

In order to determine the differential invariants of order s , we need to calculate the prolongations of the operator Y using (2.8) by considering f as a dependent variable and the variables x, y, p as independent variables:

$$Y^{(s)} = Y(x)\tilde{D}_x + Y(y)\tilde{D}_y + Y(p)\tilde{D}_p + \tilde{W}\frac{\partial}{\partial f} + \sum_{s \geq 1} \tilde{D}_{i_1} \dots \tilde{D}_{i_s}(\tilde{W}) \frac{\partial}{\partial f_{i_1 i_2 \dots i_s}}, \quad (2.23)$$

$$i_1, i_2, \dots, i_s \in \{x, y, p\},$$

where

$$\tilde{D}_k = \partial_k + f_k \partial_f + f_{ki} \partial_{f_i} + f_{kij} \partial_{f_{ij}} + \dots, \quad i, j, k \in \{x, y, p\}. \quad (2.24)$$

in which \tilde{W} is the *Lie characteristic function*

$$\tilde{W} = \mu - Y(x)f_x - Y(y)f_y - Y(p)f_p. \quad (2.25)$$

The differential invariants are determined by the equations

$$Y^{(s)}J(x, y, p, f, f_{(1)}, f_{(2)}, \dots, f_{(s)}) = 0. \quad (2.26)$$

It should be noted that since the generator $Y^{(s)}$ contains arbitrary functions $\xi(x, y)$ and $\eta(x, y)$, the corresponding identity (2.26) leads to m linear PDEs for J , where m is the number of the arbitrary functions and their derivatives that appear in $Y^{(s)}$.

For simplicity, from here on, we denote the derivative of $f(x, y, p)$ with respect to the independent variables x, y, p as f_1, f_2, f_3 . The same notation will be used for higher-order derivatives.

Now, in order to find all the fifth order differential invariants of the third-order ODE (1.1), one can solve the invariant criterion (2.26) with $s = 5$. However, for compactness of the derived differential invariants, one can replace any partial derivative with respect to x by the total derivative with respect to x . So, we need to solve the following invariant criterion

$$\begin{aligned} Y^{(5)}J(x, y, y_1, f, f_2, f_3, f_{2,2}, f_{2,3}, f_{3,3}, f_{2,2,2}, f_{2,2,3}, f_{2,3,3}, f_{3,3,3}, f_{2,2,2,2}, f_{2,2,2,3}, f_{2,2,3,3}, f_{2,3,3,3}, f_{3,3,3,3}, \\ f_{2,2,2,2,2}, f_{2,2,2,2,3}, f_{2,2,2,3,3}, f_{2,2,3,3,3}, f_{2,3,3,3,3}, f_{3,3,3,3,3}, d_{1,1}, d_{1,2}, d_{1,3}, d_{1,4}, d_{1,5}, d_{1,6}, d_{1,7}, d_{1,8}, d_{1,9}, d_{1,10}, \\ d_{1,11}, d_{1,12}, d_{1,13}, d_{1,14}, d_{1,15}, d_{2,1}, d_{2,2}, d_{2,3}, d_{2,4}, d_{2,5}, d_{2,6}, d_{2,7}, d_{2,8}, d_{2,9}, d_{2,10}, d_{3,1}, d_{3,2}, d_{3,3}, d_{3,4}, \\ d_{3,5}, d_{3,6}, d_{4,1}, d_{4,2}, d_{4,3}, d_{5,1}) = 0 \end{aligned} \quad (2.27)$$

by prolonging the infinitesimal operator $Y^{(5)}$ to the variables $d_{i,j}$ through the infinitesimals

$Y^{(5)}d_{i,j}$, where

$$\begin{aligned}
d_{1,1} &= \dot{D}_x f, d_{1,2} = \dot{D}_x f_2, d_{1,3} = \dot{D}_x f_3, d_{1,4} = \dot{D}_x f_{2,2}, d_{1,5} = \dot{D}_x f_{2,3}, d_{1,6} = \dot{D}_x f_{3,3}, d_{1,7} = \dot{D}_x f_{2,2,2}, \\
d_{1,8} &= \dot{D}_x f_{2,2,3}, d_{1,9} = \dot{D}_x f_{2,3,3}, d_{1,10} = \dot{D}_x f_{3,3,3}, d_{1,11} = \dot{D}_x f_{2,2,2,2}, d_{1,12} = \dot{D}_x f_{2,2,2,3}, d_{1,13} = \dot{D}_x f_{2,2,3,3}, \\
d_{1,14} &= \dot{D}_x f_{2,3,3,3}, d_{1,15} = \dot{D}_x f_{3,3,3,3}, d_{2,1} = \dot{D}_x^2 f, d_{2,2} = \dot{D}_x^2 f_2, d_{2,3} = \dot{D}_x^2 f_3, d_{2,4} = \dot{D}_x^2 f_{2,2}, \\
d_{2,5} &= \dot{D}_x^2 f_{2,3}, d_{2,6} = \dot{D}_x^2 f_{3,3}, d_{2,7} = \dot{D}_x^2 f_{2,2,2}, d_{2,8} = \dot{D}_x^2 f_{2,2,3}, d_{2,9} = \dot{D}_x^2 f_{2,3,3}, d_{2,10} = \dot{D}_x^2 f_{3,3,3}, \\
d_{3,1} &= \dot{D}_x^3 f, d_{3,2} = \dot{D}_x^3 f_2, d_{3,3} = \dot{D}_x^3 f_3, d_{3,4} = \dot{D}_x^3 f_{2,2}, d_{3,5} = \dot{D}_x^3 f_{2,3}, d_{3,6} = \dot{D}_x^3 f_{3,3}, \\
d_{4,1} &= \dot{D}_x^4 f, d_{4,2} = \dot{D}_x^4 f_2, d_{4,3} = \dot{D}_x^4 f_3, d_{5,1} = \dot{D}_x^5 f
\end{aligned} \tag{2.28}$$

Definition 2.6. An invariant differentiation operator of a class of second-order ODEs (1.1) is a differential operator \mathcal{D} which satisfies that if I is a differential invariant of ODE (1.1), then $\mathcal{D}I$ is its differential invariant too.

As it is shown in [9], the number of independent invariant differentiation operators \mathcal{D} equals the number of independent variables x, y and p . The invariant differentiation operators \mathcal{D} should take the form

$$\mathcal{D} = K\tilde{D}_x + L\tilde{D}_y + M\tilde{D}_p, \tag{2.29}$$

with the coordinates K, L and M satisfying the non-homogeneous linear system

$$Y^{(5)}K = \mathcal{D}(Y(x)), \quad Y^{(5)}L = \mathcal{D}(Y(y)), \quad Y^{(5)}M = \mathcal{D}(Y(p)), \tag{2.30}$$

where K, L and M are functions of the following variables

$$\begin{aligned}
&x, y, y_1, f, f_2, f_3, f_{2,2}, f_{2,3}, f_{3,3}, f_{2,2,2}, f_{2,2,3}, f_{2,3,3}, f_{3,3,3}, f_{2,2,2,2}, f_{2,2,2,3}, f_{2,2,3,3}, f_{2,3,3,3}, f_{3,3,3,3}, \\
&f_{2,2,2,2,2}, f_{2,2,2,2,3}, f_{2,2,2,3,3}, f_{2,2,3,3,3}, f_{2,3,3,3,3}, f_{3,3,3,3,3}, d_{1,1}, d_{1,2}, d_{1,3}, d_{1,4}, d_{1,5}, d_{1,6}, d_{1,7}, d_{1,8}, d_{1,9}, d_{1,10}, \\
&d_{1,11}, d_{1,12}, d_{1,13}, d_{1,14}, d_{1,15}, d_{2,1}, d_{2,2}, d_{2,3}, d_{2,4}, d_{2,5}, d_{2,6}, d_{2,7}, d_{2,8}, d_{2,9}, d_{2,10}, d_{3,1}, d_{3,2}, d_{3,3}, d_{3,4}, \\
&d_{3,5}, d_{3,6}, d_{4,1}, d_{4,2}, d_{4,3}, d_{5,1}
\end{aligned} \tag{2.31}$$

In reality, the general solution of the system (2.30) gives both the differential invariants and the differential operators. This general solution can be found by prolonging

the infinitesimal operator $Y^{(5)}$ to the variables K, L and M through the infinitesimals $Y^{(5)}K, Y^{(5)}L$ and $Y^{(5)}M$ respectively. Then solving the invariant criterion

$$Y^{(5)}J(x, y, y_1, f, f_2, f_3, f_{2,2}, f_{2,3}, f_{3,3}, f_{2,2,2}, f_{2,2,3}, f_{2,3,3}, f_{3,3,3}, f_{2,2,2,2}, f_{2,2,2,3}, f_{2,2,3,3}, f_{2,3,3,3}, f_{3,3,3,3}, f_{2,2,2,2,2}, f_{2,2,2,2,3}, f_{2,2,2,3,3}, f_{2,2,3,3,3}, f_{2,3,3,3,3}, f_{3,3,3,3,3}, d_{1,1}, d_{1,2}, d_{1,3}, d_{1,4}, d_{1,5}, d_{1,6}, d_{1,7}, d_{1,8}, d_{1,9}, d_{1,10}, d_{1,11}, d_{1,12}, d_{1,13}, d_{1,14}, d_{1,15}, d_{2,1}, d_{2,2}, d_{2,3}, d_{2,4}, d_{2,5}, d_{2,6}, d_{2,7}, d_{2,8}, d_{2,9}, d_{2,10}, d_{3,1}, d_{3,2}, d_{3,3}, d_{3,4}, d_{3,5}, d_{3,6}, d_{4,1}, d_{4,2}, d_{4,3}, d_{5,1}, K, L, M) = 0 \quad (2.32)$$

gives the implicit solution of the variables K, L and M with the differential invariants.

In this paper, we are interested in finding the fifth order differential invariants and differential operators of the general class $y'' = f(x, y, y')$ under point transformations (2.13). So, according to the theory of invariants of infinite transformation groups [9], the invariant criterion (2.32) should be split by the functions $\xi(x, y)$ and $\eta(x, y)$ and their derivatives. This gives rise to a homogeneous linear system of partial differential equations (PDEs):

$$X_i J = 0, \quad T_i J = 0, \quad i = 1 \dots 36, \quad (2.33)$$

where $X_i, i = 1 \dots 36$, are the differential operators corresponding to the coefficients of the following derivatives of $\eta(x, y)$ up to the seven order in the invariant criterion

$$\begin{aligned} &\eta, \eta_1, \eta_2, \eta_{1,1}, \eta_{1,2}, \eta_{2,2}, \eta_{1,1,1}, \eta_{1,1,2}, \eta_{1,2,2}, \eta_{2,2,2}, \eta_{1,1,1,1}, \eta_{1,1,1,2}, \eta_{1,1,2,2}, \eta_{1,2,2,2}, \eta_{2,2,2,2}, \eta_{1,1,1,1,1}, \eta_{1,1,1,1,2}, \\ &\eta_{1,1,1,2,2}, \eta_{1,1,2,2,2}, \eta_{1,2,2,2,2}, \eta_{2,2,2,2,2}, \eta_{1,1,1,1,1,1}, \eta_{1,1,1,1,1,2}, \eta_{1,1,1,1,2,2}, \eta_{1,1,1,2,2,2}, \eta_{1,1,2,2,2,2}, \eta_{1,2,2,2,2,2}, \eta_{2,2,2,2,2,2}, \\ &\eta_{1,1,1,1,1,1,1}, \eta_{1,1,1,1,1,1,2}, \eta_{1,1,1,1,1,2,2}, \eta_{1,1,1,1,2,2,2}, \eta_{1,1,1,2,2,2,2}, \eta_{1,1,2,2,2,2,2}, \eta_{1,2,2,2,2,2,2}, \eta_{2,2,2,2,2,2,2} \end{aligned} \quad (2.34)$$

and $T_i, i = 1 \dots 36$, are the differential operators corresponding to the coefficients of the following derivatives of $\xi(x, y)$ up to the seven order in the invariant criterion

$$\begin{aligned} &\xi, \xi_1, \xi_2, \xi_{1,1}, \xi_{1,2}, \xi_{2,2}, \xi_{1,1,1}, \xi_{1,1,2}, \xi_{1,2,2}, \xi_{2,2,2}, \xi_{1,1,1,1}, \xi_{1,1,1,2}, \xi_{1,1,2,2}, \xi_{1,2,2,2}, \xi_{2,2,2,2}, \xi_{1,1,1,1,1}, \xi_{1,1,1,1,2}, \\ &\xi_{1,1,1,2,2}, \xi_{1,1,2,2,2}, \xi_{1,2,2,2,2}, \xi_{2,2,2,2,2}, \xi_{1,1,1,1,1,1}, \xi_{1,1,1,1,1,2}, \xi_{1,1,1,1,2,2}, \xi_{1,1,1,2,2,2}, \xi_{1,1,2,2,2,2}, \xi_{1,2,2,2,2,2}, \xi_{2,2,2,2,2,2}, \\ &\xi_{1,1,1,1,1,1,1}, \xi_{1,1,1,1,1,1,2}, \xi_{1,1,1,1,1,2,2}, \xi_{1,1,1,1,2,2,2}, \xi_{1,1,1,2,2,2,2}, \xi_{1,1,2,2,2,2,2}, \xi_{1,2,2,2,2,2,2}, \xi_{2,2,2,2,2,2,2} \end{aligned} \quad (2.35)$$

Functionally independent solutions of the system (2.33) provide all independent differential invariants of $y'' = f(x, y, y')$ up to the fifth order under point transformations, as well as an implicit solution of the variables K, L and M which provide the differential

operators via (2.29). The solution of system (2.33) is found in many steps using Maple as follows:

First, let us consider the subsystem induced by the sixth and seventh derivatives of ξ and η

$$X_i J = 0, \quad T_i J = 0, \quad i = 22 \dots 36. \quad (2.36)$$

where the operators X_i and T_i , $i = 22 \dots 36$ are given in Appendix A in term of the variables $z_i, i = 1 \dots 62$ after relabeling the variables

$$\begin{aligned} & x, y, y_1, f, f_2, f_3, f_{2,2}, f_{2,3}, f_{3,3}, f_{2,2,2}, f_{2,2,3}, f_{2,3,3}, f_{3,3,3}, f_{2,2,2,2}, f_{2,2,2,3}, f_{2,2,3,3}, f_{2,3,3,3}, f_{3,3,3,3}, \\ & f_{2,2,2,2,2}, f_{2,2,2,2,3}, f_{2,2,2,3,3}, f_{2,2,3,3,3}, f_{2,3,3,3,3}, f_{3,3,3,3,3}, d_{1,1}, d_{1,2}, d_{1,3}, d_{1,4}, d_{1,5}, d_{1,6}, d_{1,7}, d_{1,8}, d_{1,9}, d_{1,10}, \\ & d_{1,11}, d_{1,12}, d_{1,13}, d_{1,14}, d_{1,15}, d_{2,1}, d_{2,2}, d_{2,3}, d_{2,4}, d_{2,5}, d_{2,6}, d_{2,7}, d_{2,8}, d_{2,9}, d_{2,10}, d_{3,1}, d_{3,2}, d_{3,3}, d_{3,4}, \\ & d_{3,5}, d_{3,6}, d_{4,1}, d_{4,2}, d_{4,3}, d_{5,1}, K, L, M \end{aligned} \quad (2.37)$$

by the variables $z_i, i = 1 \dots 62$, respectively.

In 62-dimensional space of variables $z_i, i = 1 \dots 62$, the rank of the system (2.36) is 16, so it has 46 functionally independent solutions which are given as:

$$\begin{aligned} & l_1 = z_1, l_2 = z_2, l_3 = z_3, l_4 = z_4, l_5 = z_5, l_6 = z_6, l_7 = z_7, l_8 = z_8, l_9 = z_9, l_{10} = z_{10}, l_{11} = z_{11}, l_{12} = z_{12}, l_{13} = z_{13}, \\ & l_{14} = z_{15}, l_{15} = z_{16}, l_{16} = z_{17}, l_{17} = z_{18}, l_{18} = z_{21}, l_{19} = z_{22}, l_{20} = z_{23}, l_{21} = z_{24}, l_{22} = z_{25}, l_{23} = z_{26}, l_{24} = z_{27}, \\ & l_{25} = z_{28}, l_{26} = z_{29}, l_{27} = z_{30}, l_{28} = z_{32}, l_{29} = z_{33}, l_{30} = z_{34}, l_{31} = z_{37}, l_{32} = z_{38}, l_{33} = z_{39}, l_{34} = z_{40}, l_{35} = z_{41}, \\ & l_{36} = z_{42}, l_{37} = z_{44}, l_{38} = z_{45}, l_{39} = z_{48}, l_{40} = z_{49}, l_{41} = z_{50}, l_{42} = z_{52}, l_{43} = z_{55}, l_{44} = z_{60}, l_{45} = z_{61}, l_{46} = z_{62} \end{aligned} \quad (2.38)$$

Second, let us consider the subsystem induced by the fifth derivatives of ξ and η

$$X_i J = 0, \quad T_i J = 0, \quad i = 16 \dots 21. \quad (2.39)$$

where the inherited operators X_i and T_i , $i = 16 \dots 21$ are given in Appendix B in term of the new variables $l_i, i = 1 \dots 46$.

In 46-dimensional space of variables $l_i, i = 1 \dots 46$, the rank of the system (2.39) is 10, so

it has 36 functionally independent solutions which are given as:

$$\begin{aligned}
m_1 &= l_1, m_2 = l_2, m_3 = l_3, m_4 = l_4, m_5 = l_5, m_6 = l_6, m_7 = l_7, m_8 = l_8, m_9 = l_9, \\
m_{10} &= l_{11}, m_{11} = l_{12}, m_{12} = l_{13}, m_{13} = l_{15}, m_{14} = l_{16}, m_{15} = l_{17}, m_{16} = l_{19}, m_{17} = l_{20}, \\
m_{18} &= l_{21}, m_{19} = l_{22}, m_{20} = l_{23}, m_{21} = l_{24}, m_{22} = l_{26}, m_{23} = l_{27}, m_{24} = l_{29}, m_{25} = l_{30}, \\
m_{26} &= l_{32}, m_{27} = l_{33}, m_{28} = l_{34}, m_{29} = l_{36}, m_{30} = l_{38}, m_{31} = -4l_{28} + 6l_{10} + l_{39}, m_{32} = l_{40}, \\
m_{33} &= -4l_{37} + 6l_{25} + l_{43}, m_{34} = l_{44}, m_{35} = l_{45}, m_{36} = l_{46}
\end{aligned} \tag{2.40}$$

Third, let us consider the subsystem induced by the fourth derivatives of ξ and η

$$X_i J = 0, \quad T_i J = 0, \quad i = 11 \dots 15. \tag{2.41}$$

where the inherited operators X_i and T_i , $i = 11 \dots 15$ are given in Appendix C in term of the new variables $m_i, i = 1 \dots 36$.

In 36-dimensional space of variables $m_i, i = 1 \dots 36$, the rank of the system (2.41) is 10, so it has 26 functionally independent solutions which are given as:

$$\begin{aligned}
n_1 &= m_1, n_2 = m_2, n_3 = m_3, n_4 = m_4, n_5 = m_5, n_6 = m_6, n_7 = m_8, n_8 = m_9, n_9 = m_{11}, \\
n_{10} &= m_{12}, n_{11} = m_{14}, n_{12} = m_{15}, n_{13} = m_{17}, n_{14} = m_{18}, n_{15} = m_{19}, n_{16} = m_{21}, n_{17} = m_{23}, \\
n_{18} &= m_{25}, n_{19} = m_{27}, n_{20} = 6m_7 - 4m_{22} + m_{30}, n_{21} = -3m_9m_7 + m_{12}m_{20} - m_6m_{24} + \\
&4m_6m_{10} + m_{31}, n_{22} = -2m_{24} + 2m_{10} + m_{32}, n_{23} = 6m_6m_7 - 3m_9m_{20} + m_{33}, \\
n_{24} &= m_{34}, n_{25} = m_{35}, n_{26} = m_{36}
\end{aligned} \tag{2.42}$$

Fourth, let us consider the subsystem induced by the third derivatives of ξ and η

$$X_i J = 0, \quad T_i J = 0, \quad i = 7 \dots 10. \tag{2.43}$$

where the inherited operators X_i and T_i , $i = 7 \dots 10$ are given in Appendix D in term of the new variables $n_i, i = 1 \dots 26$.

In 26-dimensional space of variables $n_i, i = 1 \dots 26$, the rank of the system (2.43) is 8, so

it has 18 functionally independent solutions which are given as:

$$\begin{aligned}
t_1 &= n_1, t_2 = n_2, t_3 = n_3, t_4 = n_4, t_5 = n_6, t_6 = n_8, t_7 = n_{10}, t_8 = n_{12}, t_9 = n_{13}, t_{10} = n_{14}, \\
t_{11} &= n_{19}, t_{12} = -3n_8n_5 - n_6n_{17} + 4n_7n_6 + n_{20}, t_{13} = 4n_7^2 - n_{17}n_7 + 2n_5n_{18} - 6n_5n_9 + n_{21}, \\
t_{14} &= -2n_{10}n_5 - n_8n_{17} + n_7n_8 + n_{16}n_{10} + n_6n_{18} + n_{22}, t_{15} = -3n_6n_8n_5 - n_{17}n_6^2 - 3n_{17}n_5 \\
&+ 4n_6^2n_7 + 4n_{16}n_7 - n_{16}n_{17} + n_{23}, t_{16} = n_{24}, t_{17} = n_{25}, t_{18} = n_{26}
\end{aligned} \tag{2.44}$$

Finally, let us consider the subsystem induced by the zero, first and second derivatives of ξ and η

$$X_i J = 0, \quad T_i J = 0, \quad i = 1 \dots 6. \tag{2.45}$$

where the inherited operators X_i and T_i , $i = 1 \dots 6$ are given in Appendix E in term of the new variables t_i , $i = 1 \dots 18$ which can be rewritten, by backing substitution, as

$$\begin{aligned}
t_1 &= x, t_2 = y, t_3 = y_1, t_4 = f, t_5 = f_3, t_6 = f_{3,3}, t_7 = f_{3,3,3}, t_8 = f_{3,3,3,3}, \\
t_9 &= f_{2,3,3,3,3}, t_{10} = f_{3,3,3,3,3}, t_{11} = \dot{D}_x f_{3,3,3,3,3}, t_{16} = K, t_{17} = L, t_{18} = M
\end{aligned} \tag{2.46}$$

and

$$\begin{aligned}
t_{12} &= 4f_3f_{2,3} - f_3\dot{D}_x f_{3,3} - 3f_{3,3}f_2 + 6f_{2,2} + \dot{D}_x^2 f_{3,3} - 4\dot{D}_x f_{2,3}, \\
t_{13} &= \tilde{D}_y t_{12} \\
t_{14} &= \tilde{D}_p t_{12}, \\
t_{15} &= f_3 t_{12} + \dot{D}_x t_{12}
\end{aligned} \tag{2.47}$$

It should be noted here that t_8 and t_{12} are the fourth order Tresse relative invariants. It is well known that a scalar second-order ODE is linearizable via a point transformation if and only if they both vanish identically as shown by Theorem 1.1. Moreover, it is noted that t_{13}, t_{14} and t_{15} vanish identically when $t_{12} = 0$.

One can see that the operators X_i and T_i , $i = 1 \dots 6$ form a Lie algebra \mathcal{L}_{12} with the

nonzero commutators

$$\begin{aligned}
[X_2, X_3] &= X_2, & [X_2, X_5] &= 2 X_4, & [X_2, X_6] &= X_5, & [X_2, T_2] &= -X_2, \\
[X_2, T_3] &= T_2 - X_3, & [X_2, T_4] &= -X_4, & [X_2, T_5] &= -X_5 + 2 T_4, & [X_2, T_6] &= -X_6 + T_5, \\
[X_3, X_4] &= -X_4, & [X_3, X_6] &= X_6, & [X_3, T_3] &= T_3, & [X_3, T_5] &= T_5, \\
[X_3, T_6] &= 2 T_6, & [X_4, T_2] &= -2 X_4, & [X_4, T_3] &= T_4 - X_5, & [X_5, T_2] &= -X_5, \\
[X_5, T_3] &= -2 X_6 + T_5, & [X_6, T_3] &= T_6, & [T_2, T_3] &= -T_3, & [T_2, T_4] &= T_4, \\
[T_2, T_6] &= -T_6, & [T_3, T_4] &= T_5, & [T_3, T_5] &= 2 T_6
\end{aligned} \tag{2.48}$$

Moreover, the projection of the operators X_i and T_i , $i = 1 \dots 6$ on the 4-dimensional space of variables $t_i, i = 1 \dots 4$ are the generators of the original infinite Lie algebra spanned by the infinitesimal operators (2.15) before the prolongation to the fifth order.

In section 4, the joint invariants of the operators (2.45) provide all differential invariants of $y'' = f(x, y, y')$, with $f_{3,3,3,3} \neq 0$, up to the fifth order under point transformations.

3 The infinitesimal point equivalence transformations

In order to find continuous group of equivalence transformations of the class (1.1) we consider the arbitrary function f that appears in our equation as a dependent variable and the variables $x, y, y' = p$ as independent variables and apply the Lie infinitesimal invariance criterion [9], that is we look for the infinitesimal ξ, η and μ of the equivalence operator Y :

$$Y = \xi(x, y)\partial_x + \eta(x, y)\partial_y + \mu(x, y, p, f)\partial_f, \tag{3.49}$$

such that its prolongation leaves the equation (1.1) invariant.

The prolongation of operator Y can be given using (2.8) as

$$Y = \xi(x, y)D_x + W\partial_y + D_x(W)\partial_p + D_x^2(W)\partial_{y''} + \mu(x, y, p, f)\partial_f, \tag{3.50}$$

where

$$D_x = \frac{\partial}{\partial x} + p \frac{\partial}{\partial y} + y'' \frac{\partial}{\partial p} + y''' \frac{\partial}{\partial y''} + \dots$$

is the operator of total derivative and $W = \eta(x, y) - \xi(x, y)p$ is the characteristic of infinitesimal operator $X = \xi(x, y)\partial_x + \eta(x, y)\partial_y$.

So, the Lie infinitesimal invariance criterion gives $\mu = \dot{D}_x^2(W) + \xi(x, y)\dot{D}_x f$ for arbitrary functions $\xi(x, y)$ and $\eta(x, y)$ where $\dot{D}_x = \frac{\partial}{\partial x} + p \frac{\partial}{\partial y} + f \frac{\partial}{\partial p}$.

Thus, equation (1.1) admits an infinite continuous group of equivalence transformations generated by the Lie algebra $\mathcal{L}_\mathcal{E}$ spanned by the following infinitesimal operators

$$U = \xi(x, y) \frac{\partial}{\partial x} - p D_x(\xi) \partial_p - (2f D_x(\xi) + p \dot{D}_x^2(\xi)) \partial_f, \quad (3.51)$$

$$V = \eta(x, y) \partial_y + D_x(\eta) \partial_p + \dot{D}_x^2(\eta) \partial_f, \quad (3.52)$$

The infinitesimal point equivalence transformations (3.51)-(3.52) can be written in the finite form as in (2.16)-(2.17), respectively, where ϕ and ψ are arbitrary functions of the indicated variables.

4 Fifth-order differential invariants and invariant equations under point transformations

In this section, we derive all the fifth-order differential invariants of the general class $y'' = f(x, y, y')$, with $f_{3,3,3,3} \neq 0$, under point transformations (2.13). Moreover, the invariant differentiation operators [9] are constructed in order to get some higher-order differential invariants from the lower-order ones. Precisely, we obtain the following theorem.

Theorem 4.1. *Every second-order ODE $y'' = f(x, y, y')$, with $f_{3,3,3,3} \neq 0$, belongs to one of two classes of equations. For the first class of equation ($\nu_1 \neq 0$), there are three fifth order differential invariants, under point transformations,*

$$\beta_1 = \nu_2^4 \nu_1^{-\frac{7}{2}}, \quad \beta_2 = \nu_3^4 \nu_1^{-\frac{11}{2}}, \quad \beta_3 = \nu_4^4 \nu_1^{-5}, \quad (4.53)$$

and three invariant differential operators

$$\begin{aligned} \mathcal{D}_1 &= (f_{3,3,3,3})^{-\frac{2}{5}} \nu_1^{\frac{1}{8}} \tilde{D}_p, \\ \mathcal{D}_2 &= (f_{3,3,3,3})^{\frac{1}{5}} \nu_1^{-\frac{3}{8}} \left(\tilde{D}_x + p \tilde{D}_y + f \tilde{D}_p \right), \\ \mathcal{D}_3 &= (f_{3,3,3,3})^{-\frac{6}{5}} \nu_1^{-\frac{1}{4}} \left(f_{3,3,3,3,3} \tilde{D}_x + (5f_{3,3,3,3} + p f_{3,3,3,3,3}) \tilde{D}_y + (10f_3 f_{3,3,3,3} + f f_{3,3,3,3,3} + 5 \dot{D}_x f_{3,3,3,3,3}) \tilde{D}_p \right), \end{aligned} \quad (4.54)$$

which satisfy the higher order relations

$$\begin{aligned} \mathcal{D}_1 \mathcal{D}_2 H - \mathcal{D}_2 \mathcal{D}_1 H - \rho_1 \mathcal{D}_1 H - \rho_2 \mathcal{D}_2 H - \rho_3 \mathcal{D}_3 H &= 0, \\ \mathcal{D}_1 \mathcal{D}_3 H - \mathcal{D}_3 \mathcal{D}_1 H - \sigma_1 \mathcal{D}_1 H - \sigma_2 \mathcal{D}_2 H - \sigma_3 \mathcal{D}_3 H &= 0, \\ \mathcal{D}_2 \mathcal{D}_3 H - \mathcal{D}_3 \mathcal{D}_2 H - \omega_1 \mathcal{D}_1 H - \omega_2 \mathcal{D}_2 H - \omega_3 \mathcal{D}_3 H &= 0, \end{aligned} \quad (4.55)$$

for any differential invariant H .

However, there is no fifth-order differential invariants for the second class ($\nu_1 = 0$), where ν_1, ν_2, ν_3 and ν_4 are the relative invariants given by (4.56) and the commutator invariants $\rho_1, \rho_2, \rho_3, \sigma_1, \sigma_2, \sigma_3$ and $\omega_1, \omega_2, \omega_3$ can be given by (4.65).

Proof. The joint invariants of the operators (2.45) provide all differential invariants of $y'' = f(x, y, y')$, with $f_{3,3,3,3} \neq 0$, up to the fifth order under point transformations, as well as an implicit solution of the variables K, L and M which provide the differential operators via (2.29).

The joint invariants of the first derived subgroup of \mathcal{L}_{12} can be given for the case $f_{3,3,3,3} \neq 0$

after backing substitution as an arbitrary function $J(x, y, \nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6, \nu_7)$, where

$$\begin{aligned}\nu_1 &= t_8^{\frac{1}{5}} t_{12}, \\ \nu_2 &= t_8^{-\frac{6}{5}} (t_{12} t_{10} + 5 t_8 t_{14}), \\ \nu_3 &= t_8^{-\frac{3}{5}} (5 t_{11} t_{12} + (7 t_5 t_{12} + t_{15}) t_8), \\ \nu_4 &= t_8^{-2} ((10 t_5 t_{14} - 5 t_{12} t_6 + 5 t_{13}) t_8^2 + ((-3 t_5 t_{10} - 5 t_9) t_{12} + t_{15} t_{10} + 5 t_{14} t_{11}) t_8)\end{aligned}\tag{4.56}$$

and

$$\begin{aligned}\nu_5 &= f_{3,3,3,3}^{\frac{1}{5}} (L - K y_1), \\ \nu_6 &= \frac{1}{5} f_{3,3,3,3}^{-\frac{6}{5}} (5 K f_{3,3,3,3} + K y_1 f_{3,3,3,3,3} - f_{3,3,3,3,3} L), \\ \nu_7 &= f_{3,3,3,3}^{-\frac{3}{5}} \left(2 f_{3,3,3,3} K f_3 y_1 - f_{3,3,3,3} K f - 2 f_{3,3,3,3} f_3 L + f_{3,3,3,3} M + \dot{D}_x f_{3,3,3,3} (K y_1 - L) \right).\end{aligned}\tag{4.57}$$

The non-zero inheritance of the operators X_i and T_i , $i = 1 \dots 12$ in term of the new variables $x, y, \nu_i, i = 1 \dots 7$ is

$$\begin{aligned}X_1 &= [0, 1, 0, 0, 0, 0, 0, 0, 0], \\ T_1 &= [1, 0, 0, 0, 0, 0, 0, 0, 0], \\ X_3 &= [0, 0, -\frac{8}{5} \nu_1, -\frac{7}{5} \nu_2, -\frac{11}{5} \nu_3, -2 \nu_4, \frac{2}{5} \nu_5, \frac{3}{5} \nu_6, -\frac{1}{5} \nu_7], \\ T_2 &= X_3.\end{aligned}\tag{4.58}$$

The joint invariants of the operators (4.58) are the invariants of the operator

$$Z = 8 \nu_1 \frac{\partial}{\partial \nu_1} + 7 \nu_2 \frac{\partial}{\partial \nu_2} + 11 \nu_3 \frac{\partial}{\partial \nu_3} + 10 \nu_4 \frac{\partial}{\partial \nu_4} - 2 \nu_5 \frac{\partial}{\partial \nu_5} - 3 \nu_6 \frac{\partial}{\partial \nu_6} + \nu_7 \frac{\partial}{\partial \nu_7}.\tag{4.59}$$

The invariants of the operators (4.59) can be given using characteristic method for two classes as follows:

(1) First class of equation ($\nu_1 \neq 0$)

$$\beta_1 = \nu_2^4 \nu_1^{-\frac{7}{2}}, \quad \beta_2 = \nu_3^4 \nu_1^{-\frac{11}{2}}, \quad \beta_3 = \nu_4^4 \nu_1^{-5},\tag{4.60}$$

and

$$\gamma_1 = \nu_5^8 \nu_1^2, \quad \gamma_2 = \nu_6^8 \nu_1^3, \quad \gamma_3 = \frac{\nu_7^8}{\nu_1}\tag{4.61}$$

(2) Second class of equation ($\nu_1 = 0$) does not have fifth-order differential invariants independent from the variables K, L and M . This because of vanishing the variables t_{13}, t_{14} and t_{15} identically when $t_{12} = 0$, and so $\nu_2 = \nu_3 = \nu_4 = 0$ whenever $\nu_1 = 0$.

Regarding the invariant differentiation operators, γ_1, γ_2 and γ_3 are the only invariants depending on the variables K, L and M . Then the general solution of (2.30) can be given implicitly as

$$\gamma_1 = F_1, \gamma_2 = F_2, \gamma_3 = F_3, \quad (4.62)$$

where F_1, F_2 and F_3 are the arbitrary functions of differential invariants β_i , $i = 1 \dots 3$.

Solving system (4.62) gives the variables K, L and M in terms of three arbitrary functions F_1, F_2 and F_3 which provide three independent invariant differentiation operators $\mathcal{D}_1, \mathcal{D}_2$ and \mathcal{D}_3 via (2.29).

Finally, since the matrix

$$A = \begin{pmatrix} \mathcal{D}_1 x & \mathcal{D}_2 x & \mathcal{D}_3 x \\ \mathcal{D}_1 y & \mathcal{D}_2 y & \mathcal{D}_3 y \\ \mathcal{D}_1 p & \mathcal{D}_2 p & \mathcal{D}_3 p \end{pmatrix} \quad (4.63)$$

is an invertible matrix with the non-zero determinant $J = 5 f_{3,3,3,3}^{-\frac{2}{5}} \nu_1^{-\frac{1}{2}}$, then the invariant differential operators should satisfy the commutation relations

$$\begin{aligned} [\mathcal{D}_1, \mathcal{D}_2] &= \rho_1 \mathcal{D}_1 + \rho_2 \mathcal{D}_2 + \rho_3 \mathcal{D}_3, \\ [\mathcal{D}_1, \mathcal{D}_3] &= \sigma_1 \mathcal{D}_1 + \sigma_2 \mathcal{D}_2 + \sigma_3 \mathcal{D}_3, \\ [\mathcal{D}_2, \mathcal{D}_3] &= \omega_1 \mathcal{D}_1 + \omega_2 \mathcal{D}_2 + \omega_3 \mathcal{D}_3, \end{aligned} \quad (4.64)$$

where

$$\begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix} = A^{-1} \begin{pmatrix} \mathcal{D}_1 \mathcal{D}_2 x - \mathcal{D}_2 \mathcal{D}_1 x \\ \mathcal{D}_1 \mathcal{D}_2 y - \mathcal{D}_2 \mathcal{D}_1 y \\ \mathcal{D}_1 \mathcal{D}_2 p - \mathcal{D}_2 \mathcal{D}_1 p \end{pmatrix}, \quad \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} = A^{-1} \begin{pmatrix} \mathcal{D}_1 \mathcal{D}_3 x - \mathcal{D}_3 \mathcal{D}_1 x \\ \mathcal{D}_1 \mathcal{D}_3 y - \mathcal{D}_3 \mathcal{D}_1 y \\ \mathcal{D}_1 \mathcal{D}_3 p - \mathcal{D}_3 \mathcal{D}_1 p \end{pmatrix},$$

$$\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = A^{-1} \begin{pmatrix} \mathcal{D}_2 \mathcal{D}_3 x - \mathcal{D}_3 \mathcal{D}_2 x \\ \mathcal{D}_2 \mathcal{D}_3 y - \mathcal{D}_3 \mathcal{D}_2 y \\ \mathcal{D}_2 \mathcal{D}_3 p - \mathcal{D}_3 \mathcal{D}_2 p \end{pmatrix}. \quad (4.65)$$

Hence the commutator identities (4.64) can be applied to any differential invariants H to give the higher order relations (4.55). \square

5 Application

In this section, invariant description of all the canonical forms in the complex plane [6] for second-order ODEs $y'' = f(x, y, y')$ where both of the two Tressé relative invariants (1.2) are non-zero is provided. Moreover, one example of the second class ($\nu_1 = 0$) is given from the canonical forms of second order ODE in the real plane [5].

Example 5.1. Consider the canonical form of second order ODE in the complex plane with three infinitesimal symmetries [6]

$$y'' = c \exp(-y'). \quad (5.66)$$

It is an equation of the first class ($\nu_1 \neq 0$), with the three constant fifth-order differential invariants

$$\beta_1 = -65536, \beta_2 = -65536, \beta_3 = 2825761. \quad (5.67)$$

Example 5.2. Consider the canonical form of second order ODE in the complex plane

with three infinitesimal symmetries [6]

$$y'' = c y'^{\left(\frac{\alpha-2}{\alpha-1}\right)}, \quad \alpha \neq 0, \frac{1}{2}, 1, 2. \quad (5.68)$$

It is an equation of the first class ($\nu_1 \neq 0$), with the three fifth-order differential invariants

$$\beta_1 = -4096 \frac{(\alpha+1)^4}{2\alpha^3-5\alpha^2+2\alpha}, \quad \beta_2 = -4096 \frac{(\alpha+1)^4}{2\alpha^3-5\alpha^2+2\alpha}, \quad \beta_3 = \frac{(14\alpha^2+13\alpha+14)^4}{\alpha^2(2\alpha-1)^2(\alpha-2)^2}. \quad (5.69)$$

As a special case, when $\alpha = -1$, one have the second order ODE

$$y'' = c y'^{\frac{3}{2}}. \quad (5.70)$$

with the three fifth-order differential invariants

$$\beta_1 = 0, \quad \beta_2 = 0, \quad \beta_3 = 625. \quad (5.71)$$

Example 5.3. Consider the canonical form of second order ODE in the complex plane with three infinitesimal symmetries [6]

$$y'' = 6 yy' - 4 y^3 + c (y' - y^2)^{\frac{3}{2}}, \quad c \neq \pm 4i. \quad (5.72)$$

It is an equation of the first class ($\nu_1 \neq 0$), with the three fifth-order differential invariants

$$\beta_1 = 0, \quad \beta_2 = 0, \quad \beta_3 = 625 \frac{c^2}{16+c^2}. \quad (5.73)$$

Example 5.4. Consider the canonical form of second order ODE in the complex plane with three infinitesimal symmetries [6]

$$y'' = 6 yy' - 4 y^3 + c (y' - y^2)^{\frac{3}{2}}, \quad c = \pm 4i. \quad (5.74)$$

It is an equation of the second class ($\nu_1 = 0$), so it does not have fifth-order differential invariants.

Example 5.5. Consider the canonical form of second order ODE in the real plane [5]

$$x y'' = y' + y'^3 + (1 + y'^2)^{\frac{3}{2}}. \quad (5.75)$$

It is an equation of the second class ($\nu_1 = 0$), so it does not have fifth-order differential invariants.

Example 5.6. Consider the canonical form of second order ODE in the complex plane with two infinitesimal symmetries [6]

$$y'' = f(y'). \quad (5.76)$$

It is an equation of the first class ($\nu_1 \neq 0$). It has three non-constant fifth-order differential invariants. However, this class can be characterized by the relation $\beta_1 + \beta_2 = 0$ and the Jacobian matrix $\frac{\partial(\beta_1, \beta_2, \beta_3)}{\partial(x, y, p)}$ where it has rank one.

Example 5.7. Consider the canonical form of second order ODE in the complex plane with two infinitesimal symmetries [6]

$$y'' = y' + f(y' - y). \quad (5.77)$$

For the case ($\nu_1 \neq 0$), it has three non-constant fifth-order differential invariants. However, this class can be characterized by the relation $\beta_1 + \beta_2 \neq 0$ and the Jacobian matrix $\frac{\partial(\beta_1, \beta_2, \beta_3)}{\partial(x, y, p)}$ where it has rank one.

Example 5.8. Consider the canonical form of second order ODE in the complex plane with one infinitesimal symmetries [6]

$$y'' = f(x, y'). \quad (5.78)$$

For the case ($\nu_1 \neq 0$), it has three non-constant fifth-order differential invariants. However, this class can be characterized by the Jacobian matrix $\frac{\partial(\beta_1, \beta_2, \beta_3)}{\partial(x, y, p)}$ where it has rank two.

6 Conclusion

The paper provides an extension of the work of Bagderina [1] who solved the equivalence problem for scalar second-order ordinary differential equations (ODEs), cubic in the first-order derivative, via point transformations. However, the question is open for the general

class $y'' = f(x, y, y')$ which is not cubic in the first-order derivative. Lie's infinitesimal method was utilized to study the differential invariants of this general class under an arbitrary point equivalence transformations. All fifth order differential invariants and the invariant differentiation operators were determined. These are stated as Theorems 4.1 in Section 4.

As an application, the symmetry algebra of the second order ODE $y'' = f(x, y, y')$ where both of the two Tressé relative invariants (1.2) are non-zero is characterized as follows:

- 1) The symmetry algebra is 3-dimensional iff the rank of the Jacobian matrix $\frac{\partial(\beta_1, \beta_2, \beta_3)}{\partial(x, y, p)}$ is zero (the differential invariants β_1, β_2 and β_3 are constant).
- 2) The symmetry algebra is 2-dimensional iff the rank of the Jacobian matrix $\frac{\partial(\beta_1, \beta_2, \beta_3)}{\partial(x, y, p)}$ is one.
- 3) The symmetry algebra is 1-dimensional iff the rank of the Jacobian matrix $\frac{\partial(\beta_1, \beta_2, \beta_3)}{\partial(x, y, p)}$ is two.

Moreover, invariant description of all the canonical forms in the complex plane for second-order ODEs $y'' = f(x, y, y')$ where both of the two Tressé relative invariants (1.2) are non-zero is provided.

Acknowledgments

The author would like to thank the King Fahd University of Petroleum and Minerals for its support and excellent research facilities and I also want to thank Drs. Fazal Mahomed, Hassan Azad and Tahir Mustafa for several discussions.

Appendix A: The differential operators of the homogeneous linear system of PDEs (2.36)

[illegible]

Appendix B: The differential operators of the homogeneous linear system of PDEs (2.39)

$$\begin{aligned}
X_{16} &= [0, \\
&\quad 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0] \\
X_{17} &= [0, \\
&\quad 0, 1, 0, 0, 0, 0, 0, 5l_3, 2, 0, 0, 0, 0] \\
X_{18} &= [0, 1, 0, 0, 0, 0, 0, 0, \\
&\quad 0, 4l_3, 0, 2, 0, 0, 0, 10l_3^2, 8l_3, 2, 0, 0, 0] \\
X_{19} &= [0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3l_3, 0, 0, 2, 0, 0, 0, 0, \\
&\quad 0, 0, 6l_3^2, 0, 6l_3, 0, 2, 0, 10l_3^3, 12l_3^2, 6l_3, 0, 0, 0] \\
X_{20} &= [0, 0, 0, 0, 0, 0, 0, 0, 0, 2l_3, 0, 0, 0, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3l_3^2, 0, 0, 4l_3, 0, 0, \\
&\quad 2, 0, 0, 0, 4l_3^3, 0, 6l_3^2, 0, 4l_3, 0, 5l_3^4, 8l_3^3, 6l_3^2, 0, 0, 0] \\
X_{21} &= [0, 0, 0, 0, 0, 0, 0, 0, 0, l_3^2, 0, 0, 0, 2l_3, 0, 0, 0, 2, 0, 0, 0, 0, 0, 0, l_3^3, 0, 0, 2l_3^2, 0, 0, \\
&\quad 2l_3, 0, 0, 0, l_3^4, 0, 2l_3^3, 0, 2l_3^2, 0, l_3^5, 2l_3^4, 2l_3^3, 0, 0, 0] \\
T_{16} &= [0, \\
&\quad 0, 0, 0, 0, 0, 0, -l_3, -1, 0, 0, 0, 0] \\
T_{17} &= [0, \\
&\quad 0, -l_3, 0, -1, 0, 0, 0, -5l_3^2, -7l_3, -4, 0, 0, 0] \\
T_{18} &= [0, -l_3, 0, 0, -1, 0, 0, 0, \\
&\quad 0, 0, 0, -4l_3^2, 0, -6l_3, 0, -4, 0, -10l_3^3, -18l_3^2, -18l_3, 0, 0, 0] \\
T_{19} &= [0, 0, 0, 0, 0, 0, 0, 0, 0, -l_3, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, -3l_3^2, 0, 0, -5l_3, \\
&\quad 0, 0, -4, 0, 0, 0, -6l_3^3, 0, -12l_3^2, 0, -14l_3, 0, -10l_3^4, -22l_3^3, -30l_3^2, 0, 0, 0] \\
T_{20} &= [0, 0, 0, 0, 0, 0, 0, 0, 0, -2l_3^2, 0, 0, 0, -4l_3, 0, 0, 0, -4, 0, 0, 0, 0, 0, -3l_3^3, 0, 0, \\
&\quad -7l_3^2, 0, 0, -10l_3, 0, 0, 0, -4l_3^4, 0, -10l_3^3, 0, -16l_3^2, 0, -5l_3^5, -13l_3^4, -22l_3^3, 0, 0, 0] \\
T_{21} &= [0, 0, 0, 0, 0, 0, 0, 0, 0, -l_3^3, 0, 0, 0, -3l_3^2, 0, 0, 0, -6l_3, 0, 0, 0, 0, 0, -l_3^4, 0, 0, \\
&\quad -3l_3^3, 0, 0, -6l_3^2, 0, 0, 0, -l_3^5, 0, -3l_3^4, 0, -6l_3^3, 0, -l_3^6, -3l_3^5, -6l_3^4, 0, 0, 0]
\end{aligned}$$

Appendix C: The differential operators of the homogeneous linear system of PDEs (2.41)

$$\begin{aligned}
X_{11} &= [0, 1, 0, 0, 0, 0, 0, 0, 0] \\
X_{12} &= [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 4m_3, 2, 0, -m_{12}, 0, \\
&\quad 3m_9, 0, 0, 0] \\
X_{13} &= [0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3m_3, 0, 2, 0, 0, 0, 0, 6m_3^2, 6m_3, 2, \\
&\quad 3m_9 - 3m_3m_{12}, 0, -6m_6 + 9m_3m_9, 0, 0, 0] \\
X_{14} &= [0, 0, 0, 0, 0, 0, 2m_3, 0, 0, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3m_3^2, 0, 4m_3, 0, 2, 0, 0, 0, 4m_3^3, \\
&\quad 6m_3^2, 4m_3, -6m_6 + 6m_3m_9 - 3m_3^2m_{12}, 0, -12m_6m_3 + 9m_3^2m_9, 0, 0, 0] \\
X_{15} &= [0, 0, 0, 0, 0, 0, m_3^2, 0, 0, 2m_3, 0, 0, 2, 0, 0, 0, 0, 0, 0, m_3^3, 0, 2m_3^2, 0, 2m_3, 0, 0, 0, m_3^4, \\
&\quad 2m_3^3, 2m_3^2, -6m_6m_3 + 3m_3^2m_9 - m_3^3m_{12}, 0, -6m_6m_3^2 + 3m_3^3m_9, 0, 0, 0] \\
T_{11} &= [0, -m_3, -1, 0, 0, 0, 0, 0, 0] \\
T_{12} &= [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -m_3, 0, -1, 0, 0, 0, 0, 0, -4m_3^2, -6m_3, -4, \\
&\quad m_3m_{12}, 0, -3m_3m_9, 0, 0, 0] \\
T_{13} &= [0, 0, 0, 0, 0, 0, -m_3, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, -3m_3^2, 0, -5m_3, 0, -4, 0, 0, 0, -6m_3^3, \\
&\quad -12m_3^2, -14m_3, -3m_3m_9 + 3m_3^2m_{12}, -6, 6m_6m_3 - 9m_3^2m_9, 0, 0, 0] \\
T_{14} &= [0, 0, 0, 0, 0, 0, -2m_3^2, 0, 0, -4m_3, 0, 0, -4, 0, 0, 0, 0, 0, 0, -3m_3^3, 0, -7m_3^2, 0, -10m_3, \\
&\quad 0, -6, 0, -4m_3^4, -10m_3^3, -16m_3^2, 6m_6m_3 - 6m_3^2m_9 + 3m_3^3m_{12}, -12m_3, \\
&\quad 12m_6m_3^2 - 9m_3^3m_9, 0, 0, 0] \\
T_{15} &= [0, 0, 0, 0, 0, 0, -m_3^3, 0, 0, -3m_3^2, 0, 0, -6m_3, 0, 0, -6, 0, 0, 0, -m_3^4, 0, -3m_3^3, 0, -6m_3^2, \\
&\quad 0, -6m_3, 0, -m_3^5, -3m_3^4, -6m_3^3, 6m_6m_3^2 - 3m_3^3m_9 + m_3^4m_{12}, -6m_3^2, \\
&\quad 6m_6m_3^3 - 3m_3^4m_9, 0, 0, 0]
\end{aligned}$$

Appendix D: The differential operators of the homogeneous linear system of PDEs (2.43)

$$\begin{aligned}
X_7 &= [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0] \\
X_8 &= [0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3n_3, 2, 0, 0, 0, 3n_8, -2n_{18} + 6n_9, 0, 5n_{17} - 8n_7 + 3n_8n_6, \\
&\quad 0, 0, 0] \\
X_9 &= [0, 0, 0, 0, 2n_3, 0, 2, 0, 0, 0, 0, 0, 0, 0, 3n_3^2, 4n_3, 2, 0, 0, -6n_6 + 6n_3n_8, 12n_3n_9 - 4n_3n_{18} \\
&\quad -14n_7 + 2n_{17}, 0, -6n_6^2 + 6n_8n_6n_3 - 16n_3n_7 + 10n_3n_{17} + 6n_5 - 6n_{16}, 0, 0, 0] \\
X_{10} &= [0, 0, 0, 0, n_3^2, 0, 2n_3, 0, 2, 0, 0, 0, 0, 0, n_3^3, 2n_3^2, 2n_3, 0, 0, -6n_6n_3 + 3n_3^2n_8, -2n_3^2n_{18} \\
&\quad + 6n_3^2n_9 + 2n_3n_{17} - 14n_3n_7 + 12n_5, 0, -6n_6^2n_3 + 3n_8n_6n_3^2 + 5n_3^2n_{17} - 8n_3^2n_7 \\
&\quad + 6n_3n_5 - 6n_3n_{16}, 0, 0, 0] \\
T_7 &= [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -n_3, -1, 0, 0, 0, 0, 0, n_{10}, 4n_7 - n_{17}, 0, 0, 0] \\
T_8 &= [0, 0, 0, 0, -n_3, 0, -1, 0, 0, 0, 0, 0, 0, 0, -3n_3^2, -5n_3, -4, 0, 0, -3n_3n_8, -6n_3n_9 + 2n_3n_{18} \\
&\quad + 4n_7 - n_{17}, -3n_8 + 3n_3n_{10}, -3n_8n_6n_3 + 20n_3n_7 - 8n_3n_{17} - 12n_5, 0, 0, 0] \\
T_9 &= [0, 0, 0, 0, -2n_3^2, 0, -4n_3, 0, -4, 0, 0, 0, 0, 0, -3n_3^3, -7n_3^2, -10n_3, -6, 0, 6n_6n_3 \\
&\quad - 6n_3^2n_8, -12n_3^2n_9 + 4n_3^2n_{18} - 4n_3n_{17} + 22n_3n_7 - 12n_5, 6n_6 - 6n_3n_8 + 3n_3^2n_{10}, \\
&\quad 6n_6^2n_3 - 6n_8n_6n_3^2 - 13n_3^2n_{17} + 28n_3^2n_7 - 30n_3n_5 + 6n_3n_{16}, 0, 0, 0] \\
T_{10} &= [0, 0, 0, 0, -n_3^3, 0, -3n_3^2, 0, -6n_3, 0, -6, 0, 0, 0, -n_3^4, -3n_3^3, -6n_3^2, -6n_3, 0, 6n_6n_3^2 \\
&\quad - 3n_3^3n_8, -6n_3^3n_9 + 2n_3^3n_{18} + 18n_3^2n_7 - 3n_3^2n_{17} - 24n_3n_5, -3n_3^2n_8 + 6n_6n_3 \\
&\quad + n_3^3n_{10}, 6n_6^2n_3^2 - 3n_3^3n_8n_6 + 12n_3^3n_7 + 6n_3^2n_{16} - 18n_3^2n_5 - 6n_3^3n_{17}, 0, 0, 0]
\end{aligned}$$

Appendix E: The differential operators of the homogeneous linear system of PDEs (2.45)

$$\begin{aligned}
X_1 &= [0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] \\
X_2 &= [0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, t_{16}, 0] \\
X_3 &= [0, 0, t_3, t_4, 0, -t_6, -2t_7, -3t_8, -4t_9, -4t_{10}, -3t_{11}, -t_{12}, -2t_{13}, -2t_{14}, -t_{15}, 0, t_{17}, t_{18}] \\
X_4 &= [0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, t_{16}, 0] \\
X_5 &= [0, 0, 0, 2t_3, 2, 0, 0, 0, -t_{10}, 0, -3t_8, 0, -t_{14}, 0, t_{12}, 0, 0, t_{17} + t_{16}t_3] \\
X_6 &= [0, 0, 0, t_3^2, 2t_3, 2, 0, 0, -3t_8 - t_3t_{10}, 0, -3t_3t_8, 0, -t_3t_{14} - t_{12}, 0, t_3t_{12}, 0, 0, t_{17}t_3] \\
T_1 &= [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] \\
T_2 &= [0, 0, -t_3, -2t_4, -t_5, 0, t_7, 2t_8, 2t_9, 3t_{10}, t_{11}, -2t_{12}, -2t_{13}, -t_{14}, -3t_{15}, t_{16}, 0, -t_{18}] \\
T_3 &= [0, 0, -t_3^2, -3t_4t_3, -t_5t_3 - 3t_4, -4t_5 + t_3t_6, -3t_6 + 3t_3t_7, 5t_3t_8, -t_{11} + t_4t_{10} + 6t_3t_9, \\
&\quad 5t_8 + 7t_3t_{10}, 5t_4t_8 + 4t_3t_{11}, -t_3t_{12}, t_5t_{12} + t_4t_{14} - t_{15}, t_3t_{14} - t_{12}, -4t_4t_{12} - 2t_3t_{15}, t_{17}, 0, -2t_{18}t_3] \\
T_4 &= [0, 0, 0, -t_3, -1, 0, 0, 0, 0, 0, 2t_8, 0, 0, 0, -3t_{12}, 0, 0, -t_{16}t_3] \\
T_5 &= [0, 0, 0, -2t_3^2, -4t_3, -4, 0, 0, t_3t_{10} + 2t_8, 0, 7t_3t_8, 0, t_3t_{14} - 2t_{12}, 0, -7t_3t_{12}, 0, 0, -t_{16}t_3^2 - t_{17}t_3] \\
T_6 &= [0, 0, 0, -t_3^3, -3t_3^2, -6t_3, -6, 0, t_3^2t_{10} + 5t_3t_8, 0, 5t_3^2t_8, 0, -t_3t_{12} + t_3^2t_{14}, 0, -4t_3^2t_{12}, 0, 0, -t_{17}t_3^2]
\end{aligned}$$

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